

Plancherel-Rotach Asymptotics of Second-Order Difference Equations with Linear Coefficients

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Abstract

In this paper, we provide a complete Plancherel-Rotach asymptotic analysis of polynomials that satisfy a second-order difference equation with linear coefficients. According to the signs of the parameters, we classify the difference equations into six cases and derive explicit asymptotic formulas of the polynomials in the outer and oscillatory regions, respectively. It is remarkable that the zero distributions of the polynomials may locate on the imaginary line or even on a sideways Y-shape curve in some cases.

Keywords: Difference equations; polynomials; Plancherel-Rotach Asymptotics; zero distributions.

AMS Subject Classification: 39A06; 41A60

1 Introduction

All of the classical hypergeometric (monic) orthogonal polynomials $\pi_n(x)$ within Askey scheme [6] satisfy the following second-order linear difference equation

$$\pi_{n+1}(x) = (x - A_n)\pi_n(x) - B_n\pi_{n-1}(x), \quad \pi_0(x) = 1, \quad \pi_1(x) = x - A_0, \quad (1.1)$$

where the coefficients A_n and B_n are polynomials or rational functions of n . For instance, the Charlier polynomials correspond to $A_n = n + a$ and $B_n = na$; the Hermite polynomials correspond to $A_n = 0$ and $B_n = n/2$; and the Chebyshev polynomials correspond to $A_n = 0$ and $B_n = 1/4$. In this paper, we will provide a complete Plancherel-Rotach asymptotic analysis of second-order difference equations with linear coefficients, namely, A_n and B_n are linear functions of n . Upon a shift on x , we may assume $A_n = dn$ and $B_n = an + b$.

There are plenty of methods developed for asymptotic analysis of orthogonal polynomials: if the polynomials can be expressed in terms of an integral, one may adopt the classical Laplace's method and steepest-descent method [14]; if the polynomials satisfy a second-order linear differential equation, the well-known WKB method [7] can be applied; if the polynomials have an explicit orthogonal weight with certain nice properties, we may use the Riemann-Hilbert approach and Deift-Zhou nonlinear steepest-descent method [1, 4, 5]. However, few studies in the previous literature were considering asymptotic analysis of polynomials via difference equations due to the loss of continuity. Van Assche and Geronimo [8] did some pioneer works in this field and obtained asymptotic formulas in the outer region, where trapezoidal rule was used to build a bridge from discreteness to continuity. Wong and Li [15] derived two linearly independent solutions in the oscillatory region, while determining the coefficients of the linear

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combination of the two solutions with given initial values was left as an open problem. In a series of work [10, 11, 12, 13], Wang and Wong established a beautiful lemma on Airy functions to derive uniform asymptotic formulas near the turning points. It is noted that their results were based on the assumption that the asymptotic formulas in the oscillatory region were given. Recently, Wang and Wong [9] completed this framework by introducing a matching method to determine the coefficients of linear combination of Wong-Li solutions in the oscillatory region from Van Assche-Geronimo solutions in the outer region. Therefore, a systematic method of asymptotic analysis on difference equations was formulated. This method was successively applied in the study of several indeterminate moment problems [2, 3] where only difference equations were known and thus the classical Laplace's method, steepest descent method, WKB method, Riemann-Hilbert approach and Deift-Zhou nonlinear steepest-descent method seem to be unapplicable.

To further develop the difference equation technique, we study a general second-order linear difference equation with linear coefficients. We are interested in the Plancherel-Rotach asymptotic formulas of solutions in the outer region and oscillatory region. According to the signs of the parameters d and a , we classify the equations into six cases: I.A) $d > 0$ and $a > 0$; I.B) $d > 0$ and $a < 0$; I.C) $d > 0$ and $a = 0$; II.A) $d = 0$ and $a > 0$; II.B) $d = 0$ and $a < 0$; II.C) $d = 0$ and $a = 0$. The cases with $d < 0$ can be transformed to the cases with $d > 0$ by a simple reflection. Note that the classical orthogonal polynomials (Charlier, Hermite and Chebyshev, for instance) always have nonnegative $a \geq 0$ and their zeros are always real. However, if we choose $a < 0$, as we shall see later, the zero distributions of the polynomials $\pi_n(x)$ may lie on the imaginary line (subcase II.B) or even on a sideways Y-shape curve (subcase I.B).

The rest of this paper is organized as follows. In Section 2, we focus on the case $d \neq 0$ and divide this case into three subcases according to the sign of a . In Section 3, we investigate the special case $d = 0$ and again consider three subcases $a > 0$, $a < 0$ and $a = 0$ in three subsections, respectively.

2 Case I: $d \neq 0$

Upon a transformation $x \rightarrow -x$ and $\pi_n \rightarrow (-1)^n \pi_n$, we may assume without loss of generality that $d > 0$. In the following three subsections, we shall consider three subcases $a > 0$, $a < 0$ and $a = 0$, respectively.

2.1 Subcase I.A: $a > 0$

We first state our theorem.

Theorem 2.1. *Assume $d > 0$ and $a > 0$. Let $x = ny$ and $y = d + z/\sqrt{n}$. As $n \rightarrow \infty$, for $z \in \mathcal{C} \setminus [-\sqrt{nd}, 2\sqrt{a}]$, we have*

$$\begin{aligned} \pi_n(nd + \sqrt{n}z) &\sim (n/e)^n \left(\frac{z + \sqrt{z^2 - 4a}}{2\sqrt{n}} \right)^n \left(\frac{z + \sqrt{z^2 - 4a}}{2(\sqrt{nd} + z)} \right)^{-a/d^2 - \sqrt{n}(\sqrt{nd} + z)/d} \\ &\quad \times \left(\frac{\sqrt{nd} + z}{\sqrt{z^2 - 4a}} \right)^{1/2} \times \exp \left[\frac{2a - z^2 - 4\sqrt{nd}z + (z + 4\sqrt{nd})\sqrt{z^2 - 4a}}{4d^2} \right]; \end{aligned} \quad (2.1)$$

and for z in a neighborhood of $(-2\sqrt{a}, 2\sqrt{a})$, we have

$$\begin{aligned} \pi_n(nd + \sqrt{n}z) &\sim (n/e)^n \left(\frac{\sqrt{a}}{\sqrt{n}} \right)^{-a/d^2 - \sqrt{n}z/d} (d + z/\sqrt{n})^{a/d^2 + \sqrt{n}(\sqrt{nd} + z)/d} \left(\frac{\sqrt{nd} + z}{\sqrt{4a - z^2}} \right)^{1/2} \exp \left[\frac{2a - z^2 - 4\sqrt{nd}z}{4d^2} \right] \\ &\quad \times 2 \cos \left[\left(n - \frac{a}{d^2} - \frac{\sqrt{n}(\sqrt{nd} + z)}{d} \right) \arccos \frac{z}{2\sqrt{a}} - \frac{\pi}{4} + \frac{(z + 4\sqrt{nd})\sqrt{4a - z^2}}{4d^2} \right]; \end{aligned} \quad (2.2)$$

and for z in a neighborhood of $(-\sqrt{nd}, -2\sqrt{a})$, we have

$$\begin{aligned}
\pi_n(nd + \sqrt{nz}) &\sim (n/e)^n \left(\frac{-z + \sqrt{-z - 2\sqrt{a}}\sqrt{-z + 2\sqrt{a}}}{2\sqrt{n}} \right)^{-a/d^2 - \sqrt{nz}/d} \\
&\times (d + z/\sqrt{n})^{a/d^2 + \sqrt{n}(\sqrt{nd} + z)/d} \left(\frac{\sqrt{nd} + z}{\sqrt{-z - 2\sqrt{a}}\sqrt{-z + 2\sqrt{a}}} \right)^{1/2} \\
&\times \exp \left[\frac{2a - z^2 - 4\sqrt{nd}z - (z + 4\sqrt{nd})\sqrt{-z - 2\sqrt{a}}\sqrt{-z + 2\sqrt{a}}}{4d^2} \right] \\
&\times 2 \cos[\pi(-a/d^2 - \sqrt{nz}/d - 1/2)].
\end{aligned} \tag{2.3}$$

Proof. Denote

$$\pi_n(x) = \prod_{k=1}^n w_k(x).$$

It follows that

$$w_{k+1}(x) = x - dk - \frac{ak + b}{w_k(x)}.$$

Let $x = ny$ with $y \in \mathcal{C} \setminus [0, d + 2\sqrt{a}/\sqrt{n}]$. We have as $n \rightarrow \infty$,

$$w_k(x) \sim \frac{x - dk + \sqrt{(x - dk)^2 - 4ak}}{2} \times \left\{ 1 + \frac{d}{2\sqrt{(x - dk)^2 - 4ak}} + \frac{dx - d^2k}{2[(x - dk)^2 - 4ak]} \right\}.$$

The above asymptotic formula can be obtained by successive approximation and proved rigorously by induction on k . Since $(x - dk)^2 - 4ak$ is of order $O(n)$ for any $k = 1, \dots, n$, we have

$$\ln \pi_n \sim \sum_{k=1}^n \left\{ \ln \frac{x - dk + \sqrt{(x - dk)^2 - 4ak}}{2} + \frac{d}{2\sqrt{(x - dk)^2 - 4ak}} + \frac{dx - d^2k}{2[(x - dk)^2 - 4ak]} \right\}.$$

We will use trapezoidal rule to approximate the three summations on the right-hand side of the above formula. Firstly, we obtain

$$\begin{aligned}
\sum_{k=1}^n \ln \frac{x - dk + \sqrt{(x - dk)^2 - 4ak}}{2} &= \sum_{k=1}^n \ln \frac{ny - dk + \sqrt{(ny - dk)^2 - 4ak}}{2} \\
&\sim n \ln \frac{n}{2} + n \int_0^1 \ln[y - dt + \sqrt{(y - dt)^2 - 4at/n}] dt + \frac{1}{2} \ln \frac{y - d + \sqrt{(y - d)^2 - 4a/n}}{2y}.
\end{aligned}$$

A simple integration gives

$$\begin{aligned}
&n \int_0^1 \ln[y - dt + \sqrt{(y - dt)^2 - 4at/n}] dt \\
&\sim n \left\{ t \ln[y - dt + \sqrt{(y - dt)^2 - 4at/n}] + \frac{\sqrt{(y - dt)^2 - 4at/n}}{2d} - \frac{t}{2} \right. \\
&\quad \left. - \left(\frac{a}{nd^2} + \frac{y}{d} \right) \ln[dy + 2a/n - d^2t + d\sqrt{(y - dt)^2 - 4at/n}] \right\} \Big|_0^1 \\
&\sim n \ln[y - d + \sqrt{(y - d)^2 - 4a/n}] + \frac{n}{2d} (\sqrt{(y - d)^2 - 4a/n} - y) - \frac{n}{2} \\
&\quad - \left(\frac{a}{d^2} + \frac{ny}{d} \right) \ln \frac{y - d + \sqrt{(y - d)^2 - 4a/n} + 2a/(nd)}{2y + 2a/(nd)}.
\end{aligned}$$

For the sake of convenience, we introduce a new scale: $y = d + z/\sqrt{n}$ with $z \in \mathcal{C} \setminus [-\sqrt{nd}, 2\sqrt{a}]$. It follows from the above two formulas that

$$\begin{aligned} \sum_{k=1}^n \ln \frac{x - dk + \sqrt{(x - dk)^2 - 4ak}}{2} &\sim n \ln \frac{n}{2} + n \ln \frac{z + \sqrt{z^2 - 4a}}{\sqrt{n}} + \frac{\sqrt{n}}{2d} (\sqrt{z^2 - 4a} - z) - n \\ &\quad - \left(\frac{a}{d^2} + \frac{\sqrt{n}(\sqrt{nd} + z)}{d} \right) \ln \frac{z + \sqrt{z^2 - 4a} + 2a/(\sqrt{nd})}{2(\sqrt{nd} + z + a/(\sqrt{nd}))} + \frac{1}{2} \ln \frac{z + \sqrt{z^2 - 4a}}{2(\sqrt{nd} + z)}. \end{aligned}$$

A further application of trapezoidal rule yields

$$\begin{aligned} \sum_{k=1}^n \left\{ \frac{d}{2\sqrt{(x - dk)^2 - 4ak}} + \frac{dx - d^2k}{2[(x - dk)^2 - 4ak]} \right\} &\sim \int_0^1 \frac{d}{2\sqrt{(y - dt)^2 - 4at/n}} + \frac{dy - d^2t}{2[(y - dt)^2 - 4at/n]} dt \\ &\sim \frac{1}{2} \ln \frac{2(\sqrt{nd} + z)}{z + \sqrt{z^2 - 4a}} + \frac{1}{2} \ln \frac{\sqrt{nd} + z}{\sqrt{z^2 - 4a}}. \end{aligned}$$

Adding the above two formulas gives

$$\begin{aligned} \ln \pi_n &\sim n \ln \frac{n}{2} + n \ln \frac{z + \sqrt{z^2 - 4a}}{\sqrt{n}} + \frac{\sqrt{n}}{2d} (\sqrt{z^2 - 4a} - z) - n \\ &\quad - \left(\frac{a}{d^2} + \frac{\sqrt{n}(\sqrt{nd} + z)}{d} \right) \ln \frac{z + \sqrt{z^2 - 4a} + 2a/(\sqrt{nd})}{2(\sqrt{nd} + z) + 2a/(\sqrt{nd})} + \frac{1}{2} \ln \frac{\sqrt{nd} + z}{\sqrt{z^2 - 4a}}. \end{aligned}$$

Since

$$\begin{aligned} \ln \frac{z + \sqrt{z^2 - 4a} + 2a/(\sqrt{nd})}{2(\sqrt{nd} + z) + 2a/(\sqrt{nd})} &\sim \ln \frac{z + \sqrt{z^2 - 4a}}{2(\sqrt{nd} + z)} + \frac{2a}{(\sqrt{nd})(z + \sqrt{z^2 - 4a})} \\ &\quad - \frac{2a^2}{d^2n(z + \sqrt{z^2 - 4a})^2} - \frac{a}{(\sqrt{nd})(\sqrt{nd} + z)}, \end{aligned}$$

we have

$$\begin{aligned} \ln \pi_n &\sim n \ln n + n \ln \frac{z + \sqrt{z^2 - 4a}}{2\sqrt{n}} - n - \left(\frac{a}{d^2} + \frac{\sqrt{n}(\sqrt{nd} + z)}{d} \right) \ln \frac{z + \sqrt{z^2 - 4a}}{2(\sqrt{nd} + z)} \\ &\quad + \frac{\sqrt{n}(\sqrt{z^2 - 4a} - z)}{2d} - \left[\frac{2a(\sqrt{nd} + z)}{(d^2)(z + \sqrt{z^2 - 4a})} - \frac{2a^2}{d^2(z + \sqrt{z^2 - 4a})^2} - \frac{a}{(d^2)} \right] + \frac{1}{2} \ln \frac{\sqrt{nd} + z}{\sqrt{z^2 - 4a}}. \end{aligned}$$

A simple calculation yields

$$\begin{aligned} &\frac{\sqrt{n}(\sqrt{z^2 - 4a} - z)}{2d} - \left[\frac{2a(\sqrt{nd} + z)}{(d^2)(z + \sqrt{z^2 - 4a})} - \frac{2a^2}{d^2(z + \sqrt{z^2 - 4a})^2} - \frac{a}{(d^2)} \right] \\ &= \frac{\sqrt{nd}(\sqrt{z^2 - 4a} - z)}{2d^2} - \left[\frac{(\sqrt{nd} + z)(z - \sqrt{z^2 - 4a})}{(2d^2)} - \frac{(z - \sqrt{z^2 - 4a})^2}{8d^2} - \frac{a}{(d^2)} \right] \\ &= \frac{2\sqrt{nd}(\sqrt{z^2 - 4a} - z)}{4d^2} - \left[\frac{2(\sqrt{nd} + z)(z - \sqrt{z^2 - 4a})}{(4d^2)} - \frac{(z^2 - 2a - z\sqrt{z^2 - 4a})}{4d^2} - \frac{4a}{4d^2} \right] \\ &= \frac{-z^2 + 2a - 4\sqrt{nd}z + (z + 4\sqrt{nd})\sqrt{z^2 - 4a}}{4d^2}. \end{aligned}$$

Consequently,

$$\begin{aligned} \ln \pi_n &\sim n \ln n - n + n \ln \frac{z + \sqrt{z^2 - 4a}}{2\sqrt{n}} - \left(\frac{a}{d^2} + \frac{\sqrt{n}(\sqrt{nd} + z)}{d} \right) \ln \frac{z + \sqrt{z^2 - 4a}}{2(\sqrt{nd} + z)} \\ &\quad + \frac{-z^2 + 2a - 4\sqrt{nd}z + (z + 4\sqrt{nd})\sqrt{z^2 - 4a}}{4d^2} + \frac{1}{2} \ln \frac{\sqrt{nd} + z}{\sqrt{z^2 - 4a}}. \end{aligned}$$

Recall that $x = ny$ and $y = d + z/\sqrt{n}$. For any $z \in \mathcal{C} \setminus [-\sqrt{nd}, 2\sqrt{a}]$, we have $\pi_n(nd + \sqrt{nz}) \sim \Phi_n(z)$ as $n \rightarrow \infty$, where

$$\begin{aligned} \Phi_n(z) := & (n/e)^n \left(\frac{z + \sqrt{z^2 - 4a}}{2\sqrt{n}} \right)^n \left(\frac{z + \sqrt{z^2 - 4a}}{2(\sqrt{nd} + z)} \right)^{-a/d^2 - \sqrt{n}(\sqrt{nd} + z)/d} \\ & \times \left(\frac{\sqrt{nd} + z}{\sqrt{z^2 - 4a}} \right)^{1/2} \times \exp \left[\frac{2a - z^2 - 4\sqrt{nd}z + (z + 4\sqrt{nd})\sqrt{z^2 - 4a}}{4d^2} \right]. \end{aligned}$$

This proves (2.1). Note that $\Phi_n(z)$ has a branch cut on $[-\sqrt{nd}, 2\sqrt{a}]$. We take the one-sided limits and define

$$\Phi_n^\pm(z) := \lim_{\varepsilon \rightarrow 0^+} \Phi_n(z \pm i\varepsilon), \quad z \in (-\sqrt{nd}, 2\sqrt{a}).$$

It is readily seen that $\Phi_n^\pm(z)$ can be analytically extended to a neighborhood of $(-\sqrt{nd}, 2\sqrt{a})$. Moreover, if $z = z_1 + iz_2$ with $z_1 \in (-\sqrt{nd}, 2\sqrt{a})$ and $z_2 > 0$, then $\Phi_n(z) = \Phi_n^+(z)$ and $\Phi_n^-(z)/\Phi_n^+(z)$ is exponentially small as $n \rightarrow \infty$. On the other hand, if $z = z_1 + iz_2$ with $z_1 \in (-\sqrt{nd}, 2\sqrt{a})$ and $z_2 < 0$, then $\Phi_n(z) = \Phi_n^-(z)$ and $\Phi_n^+(z)/\Phi_n^-(z)$ is exponentially small as $n \rightarrow \infty$. It follows that as $n \rightarrow \infty$, $\pi_n(nd + \sqrt{nz}) \sim \Phi_n(z) \sim \Phi_n^+(z) + \Phi_n^-(z)$ for all $z = z_1 + iz_2$ with $z_1 \in (-\sqrt{nd}, 2\sqrt{a})$ and $z_2 \neq 0$. By analytic continuity, we obtain $\pi_n(nd + \sqrt{nz}) \sim \Phi_n^+(z) + \Phi_n^-(z)$ for z in a neighborhood of $(-\sqrt{nd}, 2\sqrt{a})$. A simple calculation gives

$$\begin{aligned} \Phi_n^+(z) + \Phi_n^-(z) = & (n/e)^n \left(\frac{\sqrt{a}}{\sqrt{n}} \right)^{-a/d^2 - \sqrt{n}z/d} \\ & \times (d + z/\sqrt{n})^{a/d^2 + \sqrt{n}(\sqrt{nd} + z)/d} \left(\frac{\sqrt{nd} + z}{\sqrt{4a - z^2}} \right)^{1/2} \exp \left[\frac{2a - z^2 - 4\sqrt{nd}z}{4d^2} \right] \\ & \times 2 \cos \left[\left(n - \frac{a}{d^2} - \frac{\sqrt{n}(\sqrt{nd} + z)}{d} \right) \arccos \frac{z}{2\sqrt{a}} - \frac{\pi}{4} + \frac{(z + 4\sqrt{nd})\sqrt{4a - z^2}}{4d^2} \right] \end{aligned}$$

for z in a neighborhood of $(-2\sqrt{a}, 2\sqrt{a})$, and

$$\begin{aligned} \Phi_n^+(z) + \Phi_n^-(z) = & (n/e)^n \left(\frac{-z + \sqrt{-z - 2\sqrt{a}}\sqrt{-z + 2\sqrt{a}}}{2\sqrt{n}} \right)^{-a/d^2 - \sqrt{n}z/d} \\ & \times (d + z/\sqrt{n})^{a/d^2 + \sqrt{n}(\sqrt{nd} + z)/d} \left(\frac{\sqrt{nd} + z}{\sqrt{-z - 2\sqrt{a}}\sqrt{-z + 2\sqrt{a}}} \right)^{1/2} \\ & \times \exp \left[\frac{2a - z^2 - 4\sqrt{nd}z - (z + 4\sqrt{nd})\sqrt{-z - 2\sqrt{a}}\sqrt{-z + 2\sqrt{a}}}{4d^2} \right] \\ & \times 2 \cos \left[\pi(-a/d^2 - \sqrt{n}z/d - 1/2) \right] \end{aligned}$$

for z in a neighborhood of $(-\sqrt{nd}, -2\sqrt{a})$. This completes the proof of (2.2) and (2.3). \square

2.2 Case I.B: $a < 0$

For the case $a < 0$, we observe from numerical simulation that the zeros of π_n are not solely lying on the real line, instead, they will locate on a sideways Y-shape curve (cf. Figure 2.2). This will be theoretically justified in the following theorem.

Theorem 2.2. *Assume $d > 0$ and $a < 0$. Let $x = ny$ and $y = d + z/\sqrt{n}$. Denote $A = -a > 0$. Let Γ_A be the curve in the left-half complex plane defined by the following equation*

$$\operatorname{Re} \left\{ 2\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}} - z \ln \frac{z + \sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}}{z - \sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}} \right\} = 0. \quad (2.4)$$

It is noted that the above equation formulates a sideways V-shape curve that is symmetric about the x -axis with two end points $\pm 2i\sqrt{A}$; see Figure 2.2. Let $z_A < 0$ be the intersection of Γ_A with the negative real line. To be specific, z_A is the negative real root of the following equation

$$2\sqrt{z_A^2 + 4A} - z_A \ln \frac{z_A + \sqrt{z_A^2 + 4A}}{-z_A + \sqrt{z_A^2 + 4A}} = 0. \quad (2.5)$$

As $n \rightarrow \infty$, we have for $z \in \mathcal{C} \setminus ([-\sqrt{nd}, z_A] \cup \Gamma_A)$,

$$\begin{aligned} \pi_n(nd + \sqrt{nz}) &\sim (n/e)^n \left(\frac{z + \sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}}{2\sqrt{n}} \right)^{A/d^2 - \sqrt{nz}/d} \\ &\times (d + z/\sqrt{n})^{-A/d^2 + \sqrt{n}(\sqrt{nd}+z)/d} \left(\frac{\sqrt{nd} + z}{\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}} \right)^{1/2} \\ &\times \exp\left[\frac{-2A - z^2 - 4\sqrt{nd}z + (z + 4\sqrt{nd})\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}}{4d^2} \right]; \end{aligned} \quad (2.6)$$

and for z in a neighborhood of $(-\sqrt{nd}, z_A)$, we have

$$\begin{aligned} \pi_n(nd + \sqrt{nz}) &\sim (n/e)^n \left(\frac{-z + \sqrt{-z - 2i\sqrt{A}}\sqrt{-z + 2i\sqrt{A}}}{2\sqrt{n}} \right)^{A/d^2 - \sqrt{nz}/d} \\ &\times (d + z/\sqrt{n})^{-A/d^2 + \sqrt{n}(\sqrt{nd}+z)/d} \left(\frac{\sqrt{nd} + z}{\sqrt{-z - 2i\sqrt{A}}\sqrt{-z + 2i\sqrt{A}}} \right)^{1/2} \\ &\times \exp\left[\frac{-2A - z^2 - 4\sqrt{nd}z - (z + 4\sqrt{nd})\sqrt{-z - 2i\sqrt{A}}\sqrt{-z + 2i\sqrt{A}}}{4d^2} \right] \\ &\times 2 \cos[\pi[A/d^2 - \sqrt{nz}/d - 1/2]]; \end{aligned} \quad (2.7)$$

and for z in a neighborhood of $\dot{\Gamma}_A := \Gamma_A \setminus \{z_A, \pm 2i\sqrt{A}\}$, we have

$$\begin{aligned} \pi_n(nd + \sqrt{nz}) &\sim (\sqrt{n}/e)^n (\sqrt{nd} + z)^{-A/d^2 + \sqrt{n}(\sqrt{nd}+z)/d + 1/2} \times \exp\left[\frac{-2A - z^2 - 4\sqrt{nd}z}{4d^2} \right] \\ &\times \left\{ \frac{[(z + \sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}})/2]^{A/d^2 - \sqrt{nz}/d}}{(\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}})^{1/2}} \exp\left[\frac{(z + 4\sqrt{nd})\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}}{4d^2} \right] \right. \\ &\left. + \frac{[(z - \sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}})/2]^{A/d^2 - \sqrt{nz}/d}}{(-\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}})^{1/2}} \exp\left[\frac{-(z + 4\sqrt{nd})\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}}{4d^2} \right] \right\}. \end{aligned} \quad (2.8)$$

Proof. Similar to the proof of Theorem 2.1, we obtain for $z \in \mathcal{C} \setminus ([-\sqrt{nd}, z_A] \cup \Gamma_A)$,

$$\begin{aligned} \pi_n(nd + \sqrt{nz}) &\sim (n/e)^n \left(\frac{z + \sqrt{z^2 - 4a}}{2\sqrt{n}} \right)^{-a/d^2 - \sqrt{nz}/d} \times (d + z/\sqrt{n})^{a/d^2 + \sqrt{n}(\sqrt{nd}+z)/d} \left(\frac{\sqrt{nd} + z}{\sqrt{z^2 - 4a}} \right)^{1/2} \\ &\times \exp\left[\frac{2a - z^2 - 4\sqrt{nd}z + (z + 4\sqrt{nd})\sqrt{z^2 - 4a}}{4d^2} \right] \\ &\sim (n/e)^n \left(\frac{z + \sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}}{2\sqrt{n}} \right)^{A/d^2 - \sqrt{nz}/d} \\ &\times (d + z/\sqrt{n})^{-A/d^2 + \sqrt{n}(\sqrt{nd}+z)/d} \left(\frac{\sqrt{nd} + z}{\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}} \right)^{1/2} \\ &\times \exp\left[\frac{-2A - z^2 - 4\sqrt{nd}z + (z + 4\sqrt{nd})\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}}{4d^2} \right]. \end{aligned}$$

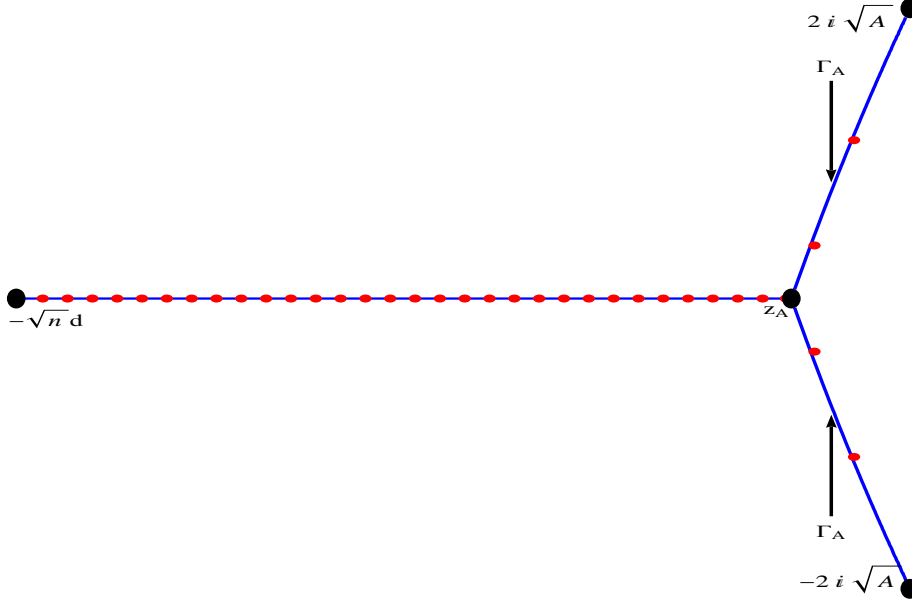


Figure 1: The sideways Y-shape branch cut (curve) and zero distribution (dots).

This gives (2.6). Denote the right-hand side of (2.6) by $\Phi_n(z)$. Note that $\Phi_n(z)$ is analytic on the complex plane except for a Y-shape branch cut $[-\sqrt{nd}, z_A] \cup \Gamma_A$ that connects $-\sqrt{nd}$ and $\pm 2i\sqrt{A}$. Moreover, $\Phi_n(z)$ is one-side continuous on the branch cut. Therefore, the functions

$$\Phi_n^\pm(z) := \lim_{\varepsilon \rightarrow 0^+} \Phi_n(z + i\varepsilon)$$

are analytic in a neighborhood of the branch cut. Note that for z in a neighborhood of $(-\infty, z_A)$,

$$\frac{\Phi_n^+(z)}{\Phi_n^-(z)} = \exp[2\pi i(A/d^2 - \sqrt{nz}/d - 1/2)];$$

and for z in a neighborhood of $\overset{\circ}{\Gamma}_A$,

$$\frac{\Phi_n^+(z)}{\Phi_n^-(z)} = -i \left(\frac{z + \sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}}{z - \sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}} \right)^{A/d^2 - \sqrt{nz}/d} \exp\left(\frac{z + 4\sqrt{nd}}{2d^2} \sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}\right).$$

It follows from the definition of Γ_A in (2.4) that the ratio Φ_n^+/Φ_n^- is exponentially large on one side and exponentially small on the other side of the branch cut $[-\sqrt{nd}, z_A] \cup \Gamma_A$. Using a similar argument in the proof of Theorem 2.1, we obtain $\pi_n(nd + \sqrt{nz}) \sim [\Phi_n^+(z) + \Phi_n^-(z)]$ for z in a neighborhood of $(-\sqrt{nd}, z_A) \cup \overset{\circ}{\Gamma}_A$. A simple calculation yields

$$\begin{aligned} \Phi_n^+(z) + \Phi_n^-(z) &= (n/e)^n \left(\frac{-z + \sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}}{2\sqrt{n}} \right)^{A/d^2 - \sqrt{nz}/d} \\ &\quad \times (d + z/\sqrt{n})^{-A/d^2 + \sqrt{n}(\sqrt{nd} + z)/d} \left(\frac{\sqrt{nd} + z}{\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}} \right)^{1/2} \\ &\quad \times \exp\left[\frac{-2A - z^2 - 4\sqrt{nd}z - (z + 4\sqrt{nd})\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}}{4d^2} \right] \\ &\quad \times 2 \cos[\pi[A/d^2 - \sqrt{nz}/d - 1/2]] \end{aligned}$$

for z in a neighborhood of $(-\sqrt{nd}, z_A)$, and

$$\begin{aligned} & \Phi_n^+(z) + \Phi_n^-(z) \\ &= (\sqrt{n}/e)^n (\sqrt{nd} + z)^{-A/d^2 + \sqrt{n}(\sqrt{nd} + z)/d + 1/2} \times \exp\left[\frac{-2A - z^2 - 4\sqrt{nd}z}{4d^2}\right] \\ & \times \left\{ \frac{[(z + \sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}})/2]^{A/d^2 - \sqrt{nz}/d}}{(\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}})^{1/2}} \exp\left[\frac{(z + 4\sqrt{nd})\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}}{4d^2}\right] \right. \\ & \left. + \frac{[(z - \sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}})/2]^{A/d^2 - \sqrt{nz}/d}}{(-\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}})^{1/2}} \exp\left[\frac{-(z + 4\sqrt{nd})\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}}{4d^2}\right] \right\} \end{aligned}$$

for z in a neighborhood of $\tilde{\Gamma}_A$. This proves (2.7) and (2.8). \square

2.3 Case I.C: $a = 0$

Theorem 2.3. Assume $d > 0$ and $a = 0$. Let $x = ny$, we have as $n \rightarrow \infty$, If y is bounded away from $[0, d]$, we have

$$\pi_n(ny) \sim (n/e)^n \left(\frac{y}{y-d}\right)^{ny/d+1/2} (y-d)^n \quad (2.9)$$

for $y \in \mathcal{C} \setminus [0, d]$; and

$$\pi_n(ny) \sim (n/e)^n (d-y)^n \left(\frac{y}{d-y}\right)^{ny/d+1/2} \times 2 \cos[\pi(n - ny/d - 1/2)]. \quad (2.10)$$

for y in a neighborhood of $(0, d)$.

Proof. Setting $z = \sqrt{n}(y-d)$ in (2.1) and taking limit $a \rightarrow 0^+$ yields (2.9). A standard argument of analytical continuity as in the proof of Theorem 2.1 gives (2.10). \square

3 Case II: $d = 0$

In this section, we consider the critical case $d = 0$. Again, we investigate three subcases according to the sign of a .

3.1 Case II.A: $a > 0$

Theorem 3.1. Assume $d = 0$ and $a > 0$. Let $x = \sqrt{ny}$. As $n \rightarrow \infty$, we have for $y \in \mathcal{C} \setminus [-2\sqrt{a}, 2\sqrt{a}]$,

$$\pi_n(\sqrt{ny}) \sim \left(\frac{n}{4e}\right)^{n/2} (y + \sqrt{y^2 - 4a})^n \left(\frac{y + \sqrt{y^2 - 4a}}{2\sqrt{y^2 - 4a}}\right)^{1/2} \left(\frac{y + \sqrt{y^2 - 4a}}{2y}\right)^{b/a} \times \exp\left[\frac{ny}{4a}(y - \sqrt{y^2 - 4a})\right]; \quad (3.1)$$

and for y in a neighborhood of $(0, 2\sqrt{a})$, we have

$$\begin{aligned} \pi_n(\sqrt{ny}) &\sim \left(\frac{na}{e}\right)^{n/2} \left(\frac{\sqrt{a}}{\sqrt{2\sqrt{a}-y}\sqrt{2\sqrt{a}+y}}\right)^{1/2} \left(\frac{\sqrt{a}}{y}\right)^{b/a} \times \exp\left[\frac{ny^2}{4a}\right] \\ &\times 2 \cos[(n + 1/2 + b/a) \arccos \frac{y}{2\sqrt{a}} - \pi/4 - \frac{ny}{4a} \sqrt{2\sqrt{a}-y}\sqrt{2\sqrt{a}+y}]; \end{aligned} \quad (3.2)$$

and for y in a neighborhood of $(-2\sqrt{a}, 0)$, we have

$$\begin{aligned} \pi_n(\sqrt{ny}) &\sim \left(\frac{na}{e}\right)^{n/2} (-1)^n \left(\frac{\sqrt{a}}{\sqrt{2\sqrt{a}-y}\sqrt{2\sqrt{a}+y}}\right)^{1/2} \left(\frac{\sqrt{a}}{-y}\right)^{b/a} \times \exp\left[\frac{ny^2}{4a}\right] \\ &\quad \times 2 \cos\left[(n+1/2+b/a) \arccos \frac{-y}{2\sqrt{a}} - \pi/4 + \frac{ny}{4a} \sqrt{2\sqrt{a}-y} \sqrt{2\sqrt{a}+y}\right] \end{aligned} \quad (3.3)$$

Proof. Denote

$$\pi_n(x) = \prod_{k=1}^n w_k(x).$$

It follows that

$$w_{k+1}(x) = x - \frac{ak+b}{w_k(x)}.$$

Let $x = \sqrt{ny}$ with $y \in \mathcal{C} \setminus [-2\sqrt{a}, 2\sqrt{a}]$. We have as $n \rightarrow \infty$,

$$w_k(x) \sim \frac{x + \sqrt{x^2 - 4ak}}{2} \times \left\{ 1 + \frac{a}{x^2 - 4ak} - \frac{2b}{(x + \sqrt{x^2 - 4ak})\sqrt{x^2 - 4ak}} \right\}.$$

By trapezoidal rule, we obtain

$$\begin{aligned} \ln \pi_n &\sim n \ln(\sqrt{n}/2) + n \int_0^1 \ln(y + \sqrt{y^2 - 4at}) dt + \frac{1}{2} \ln \frac{y + \sqrt{y^2 - 4a}}{2y} \\ &\quad + \int_0^1 \frac{a}{y^2 - 4at} dt - \frac{2b}{(y + \sqrt{y^2 - 4at})\sqrt{y^2 - 4at}} dt \\ &\sim n \ln(\sqrt{n}/2) + n \left[t \ln(y + \sqrt{y^2 - 4at}) - \frac{y}{4a} \sqrt{y^2 - 4at} - \frac{t}{2} \right] \Big|_0^1 \\ &\quad + \frac{1}{2} \ln \frac{y + \sqrt{y^2 - 4a}}{2y} + \frac{1}{4} \ln \frac{y^2}{y^2 - 4a} + \frac{b}{a} \ln(y + \sqrt{y^2 - 4a}) \Big|_0^1 \\ &\sim n \ln(\sqrt{n}/2) + n \ln(y + \sqrt{y^2 - 4a}) - \frac{ny}{4a} (\sqrt{y^2 - 4a} - y) - \frac{n}{2} \\ &\quad + \frac{1}{2} \ln \frac{y + \sqrt{y^2 - 4a}}{2y} + \frac{1}{4} \ln \frac{y^2}{y^2 - 4a} + \frac{b}{a} \ln \frac{y + \sqrt{y^2 - 4a}}{2y}. \end{aligned}$$

Recall that $x = ny$. We then obtain $\pi_n(ny) \sim \Phi_n(y)$, where

$$\Phi_n(y) := \left(\frac{n}{4e}\right)^{n/2} (y + \sqrt{y^2 - 4a})^n \left(\frac{y + \sqrt{y^2 - 4a}}{2\sqrt{y^2 - 4a}}\right)^{1/2} \left(\frac{y + \sqrt{y^2 - 4a}}{2y}\right)^{b/a} \times \exp\left[\frac{ny}{4a} (y - \sqrt{y^2 - 4a})\right].$$

By a standard argument of analytical continuity, we obtain $\pi_n(ny) \sim \Phi_n^+(y) + \phi_n^-(y)$ for y in a neighborhood of $(-2\sqrt{a}, 0) \cup (0, 2\sqrt{a})$, where

$$\Phi_n^\pm(y) := \lim_{\varepsilon \rightarrow 0^+} \Phi_n(y + i\varepsilon).$$

For y in a neighborhood of $(0, 2\sqrt{a})$, a simple calculation gives

$$\begin{aligned} \Phi_n^+(y) + \phi_n^-(y) &= \left(\frac{n}{4e}\right)^{n/2} (2\sqrt{a})^n \left(\frac{2\sqrt{a}}{2\sqrt{2\sqrt{a}-y}\sqrt{2\sqrt{a}+y}}\right)^{1/2} \left(\frac{2\sqrt{a}}{2y}\right)^{b/a} \times \exp\left[\frac{ny}{4a} (y)\right] \\ &\quad \times 2 \cos\left[(n+1/2+b/a) \arccos \frac{y}{2\sqrt{a}} - \pi/4 - \frac{ny}{4a} \sqrt{2\sqrt{a}-y} \sqrt{2\sqrt{a}+y}\right] \\ &\sim \left(\frac{na}{e}\right)^{n/2} \left(\frac{\sqrt{a}}{\sqrt{2\sqrt{a}-y}\sqrt{2\sqrt{a}+y}}\right)^{1/2} \left(\frac{\sqrt{a}}{y}\right)^{b/a} \times \exp\left[\frac{ny^2}{4a}\right] \\ &\quad \times 2 \cos\left[(n+1/2+b/a) \arccos \frac{y}{2\sqrt{a}} - \pi/4 - \frac{ny}{4a} \sqrt{2\sqrt{a}-y} \sqrt{2\sqrt{a}+y}\right]. \end{aligned}$$

Thus, (3.2) follows. Note that for $\text{Re } y < 0$, we can write

$$\begin{aligned}\Phi_n(y) &= \left(\frac{n}{4e}\right)^{n/2} (-1)^n (-y + \sqrt{-y - 2\sqrt{a}} \sqrt{-y + 2\sqrt{a}})^n \left(\frac{-y + \sqrt{-y - 2\sqrt{a}} \sqrt{-y + 2\sqrt{a}}}{2\sqrt{-y - 2\sqrt{a}} \sqrt{-y + 2\sqrt{a}}}\right)^{1/2} \\ &\quad \times \left(\frac{-y + \sqrt{-y - 2\sqrt{a}} \sqrt{-y + 2\sqrt{a}}}{-2y}\right)^{b/a} \times \exp\left[\frac{ny}{4a}(y + \sqrt{-y - 2\sqrt{a}} \sqrt{-y + 2\sqrt{a}})\right].\end{aligned}$$

It follows that for y in a neighborhood of $(-2\sqrt{a}, 0)$,

$$\begin{aligned}\Phi_n^+(y) + \phi_n^-(y) &= \left(\frac{n}{4e}\right)^{n/2} (-1)^n (2\sqrt{a})^n \left(\frac{2\sqrt{a}}{2\sqrt{2\sqrt{a} - y} \sqrt{2\sqrt{a} + y}}\right)^{1/2} \left(\frac{2\sqrt{a}}{-2y}\right)^{b/a} \times \exp\left[\frac{ny}{4a}(y)\right] \\ &\quad \times 2 \cos\left[(n + 1/2 + b/a) \arccos \frac{-y}{2\sqrt{a}} - \pi/4 + \frac{ny}{4a} \sqrt{2\sqrt{a} - y} \sqrt{2\sqrt{a} + y}\right] \\ &\sim \left(\frac{na}{e}\right)^{n/2} (-1)^n \left(\frac{\sqrt{a}}{\sqrt{2\sqrt{a} - y} \sqrt{2\sqrt{a} + y}}\right)^{1/2} \left(\frac{\sqrt{a}}{-y}\right)^{b/a} \times \exp\left[\frac{ny^2}{4a}\right] \\ &\quad \times 2 \cos\left[(n + 1/2 + b/a) \arccos \frac{-y}{2\sqrt{a}} - \pi/4 + \frac{ny}{4a} \sqrt{2\sqrt{a} - y} \sqrt{2\sqrt{a} + y}\right].\end{aligned}$$

This proves (3.3). \square

3.2 Case II.B: $a < 0$

Theorem 3.2. Assume $d = 0$ and $a < 0$. Let $x = i\sqrt{n}y$, $A = -a > 0$ and $B = -b$. As $n \rightarrow \infty$, we have for $y \in \mathcal{C} \setminus [-2\sqrt{A}, 2\sqrt{A}]$,

$$\pi_n(i\sqrt{n}y) \sim i^n \left(\frac{n}{4e}\right)^{n/2} (y + \sqrt{y^2 - 4A})^n \left(\frac{y + \sqrt{y^2 - 4A}}{2\sqrt{y^2 - 4A}}\right)^{1/2} \left(\frac{y + \sqrt{y^2 - 4A}}{2y}\right)^{B/A} \times \exp\left[\frac{ny}{4A}(y - \sqrt{y^2 - 4A})\right]; \quad (3.4)$$

and for y in a neighborhood of $(0, 2\sqrt{A})$, we have

$$\begin{aligned}\pi_n(i\sqrt{n}y) &\sim i^n \left(\frac{nA}{e}\right)^{n/2} \left(\frac{\sqrt{A}}{\sqrt{2\sqrt{A} - y} \sqrt{2\sqrt{A} + y}}\right)^{1/2} \left(\frac{\sqrt{A}}{y}\right)^{B/A} \times \exp\left[\frac{ny^2}{4A}\right] \\ &\quad \times 2 \cos\left[(n + 1/2 + B/A) \arccos \frac{y}{2\sqrt{A}} - \pi/4 - \frac{ny}{4A} \sqrt{2\sqrt{A} - y} \sqrt{2\sqrt{A} + y}\right];\end{aligned} \quad (3.5)$$

and for y in a neighborhood of $(-2\sqrt{A}, 0)$, we have

$$\begin{aligned}\pi_n(i\sqrt{n}y) &\sim i^n \left(\frac{nA}{e}\right)^{n/2} (-1)^n \left(\frac{\sqrt{A}}{\sqrt{2\sqrt{A} - y} \sqrt{2\sqrt{A} + y}}\right)^{1/2} \left(\frac{\sqrt{A}}{-y}\right)^{B/A} \times \exp\left[\frac{ny^2}{4A}\right] \\ &\quad \times 2 \cos\left[(n + 1/2 + B/A) \arccos \frac{-y}{2\sqrt{A}} - \pi/4 + \frac{ny}{4A} \sqrt{2\sqrt{A} - y} \sqrt{2\sqrt{A} + y}\right].\end{aligned} \quad (3.6)$$

Proof. The monic polynomials $p_n(z) := i^{-n} \pi_n(iz)$ satisfy the same difference equation and initial conditions of π_n with a and b replaced by $A = -a$ and $B = -b$ respectively. Theorem 3.2 follows from Theorem 3.1. \square

3.3 Case II.C: $a = 0$

Theorem 3.3. *Assume $d = 0$ and $a = 0$. As $n \rightarrow \infty$, we have for $x \in \mathcal{C} \setminus [-1, 1]$,*

$$\pi_n(x) \sim \left(\frac{x + \sqrt{x^2 - 1}}{2}\right)^{n+1} \frac{1}{\sqrt{x^2 - 1}}; \quad (3.7)$$

and for x in a neighborhood of $(-1, 1)$, we have

$$\pi_n(x) \sim \frac{\sin[(n+1) \arccos x]}{2^n \sqrt{1 - x^2}}. \quad (3.8)$$

The above asymptotic formula is actually an equality.

Proof. Note that $\pi_n(x) = U_n(x)/2^n$ with $U_n(x)$ being the Chebyshev polynomials of the second kind. Furthermore, we have for $x \in \mathcal{C} \setminus [-1, 1]$,

$$\pi_n(x) = \frac{(x + \sqrt{x^2 - 1})^{n+1} - (x - \sqrt{x^2 - 1})^{n+1}}{2^{n+1} \sqrt{x^2 - 1}}.$$

It is readily seen that $\pi_n(x) \sim \Phi_n(x)$ with

$$\Phi_n(x) := \left(\frac{x + \sqrt{x^2 - 1}}{2}\right)^{n+1} \frac{1}{\sqrt{x^2 - 1}}.$$

This proves (3.7). To be consistent, we use the argument of analytical continuity and obtain

$$\pi_n(x) \sim \lim_{\varepsilon \rightarrow 0^+} [\Phi_n(x + i\varepsilon) + \Phi_n(x - i\varepsilon)] = \frac{\sin[(n+1) \arccos x]}{2^n \sqrt{1 - x^2}}$$

for $x \in (-1, 1)$. This gives (3.8). We remark that the formula (3.8) is actually an equality. \square

References

- [1] J. Baik, T. Kriecherbauer, K. T.-R. McLaughlin, P. D. Miller, Discrete Orthogonal Polynomials. Asymptotics and Applications, Annals of Mathematics Studies, 164. Princeton University Press, Princeton, NJ, 2007. viii+170 pp.
- [2] L.-H. Cao and Y.-T. Li, Linear difference equations with a transition point at the origin, Anal. Appl. 12 (2014), 75-106.
- [3] D. Dai, M. E. H. Ismail and X.-S. Wang, Plancherel-Rotach asymptotic expansion for some polynomials from indeterminate moment problems, Constr. Approx. DOI 10.1007/s00365-013-9215-1.
- [4] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides and X. Zhou, Strong asymptotics of orthogonal polynomials with respect to exponential weights, Comm. Pure Appl. Math. 52 (1999), 1491-1552.
- [5] P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotic for the MKdV equation, Ann. of Math. 137 (1993), no. 2, 295-368.
- [6] R. Koekoek, R. F. Swarttouw, The Askey-scheme of Hypergeometric Orthogonal Polynomials and its q -analogue, Report no. 98-17, TU-Delft, 1998.
- [7] F. W. J. Olver, Asymptotics and Special Functions, Academic Press, New York, 1974. Reprinted by A. K. Peters, Wellesley, MA, 1997.

- [8] W. Van Assche and J. S. Geronimo, Asymptotics for orthogonal polynomials with regularly varying recurrence coefficients, *Rocky Mountain J. Math.* 19 (1989), no. 1, 39-49.
- [9] X.-S. Wang and R. Wong, Asymptotics of orthogonal polynomials via recurrence relations, *Analysis and Applications* 10 (2012), 215-235.
- [10] Z. Wang, Asymptotic equations for second-order linear difference equations with a turning point, Ph. D. Thesis, City University of Hong Kong, 2001.
- [11] Z. Wang and R. Wong, Uniform asymptotic expansion of $J_\nu(\nu a)$ via a difference equation, *Numer. Math.* 91 (2002), 147-193.
- [12] Z. Wang and R. Wong, Asymptotic expansions for second-order linear difference equations with a turning point, *Numer. Math.* 94 (2003), no. 1, 147-194.
- [13] Z. Wang and R. Wong, Linear difference equations with transition points, *Math. Comp.* 74 (2005), no. 250, 629-653.
- [14] R. Wong, *Asymptotic Approximations of Integrals*, Academic Press, Boston, 1989. (Reprinted by SIAM, Philadelphia, PA, 2001.)
- [15] R. Wong and H. Li, Asymptotic expansions for second-order linear difference equations, *J. Comput. Appl. Math.* 41 (1992), 65-94.